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B.A./B.Sc. FIFTH SEMESTER EXAMINATION, DECEMBER 2015
THIRD YEAR [BATCH 2013-16]
MATHEMATICS [Hons]
Paper : V

Date : 15/12/2015
Time : 11 am – 3 pm
Full Marks : 100

[Use a separate Answer Book for each Group]

Group - A

Answer **any five** questions

1. a) Let G be a group of order 8 and x be an element of G of order 4. Prove that $x^2 \in Z(G)$, the centre of G. [4]
   b) Justify the statement : "Any epimorphism of Z onto Z is an isomorphism" [3]
   c) Prove that an Abelian group with two distinct elements of order 2 must have a subgroup of order 4. [3]

2. a) Find all homomorphisms from Z_6 into Z_4. [5]
   b) Let G, G´ be two cyclic groups of order 12 and 4 respectively. Prove that there exists a homomorphism ϕ of G onto G´ with $o(\text{Ker}\,\phi) = 3$. [3]
   c) Let G be an abelian group of order 6. Find |Aut G|. [2]

3. a) Show that there are only two groups of order 6 upto isomorphism. [4]
   b) Show that D_4 can't be expressed as an internal direct product of two proper subgroups. [3]
   c) Is the group $\mathbb{Z}\times\mathbb{Z}$ cyclic? Justify your answer. [3]

4. a) State and prove Sylow's 3rd theorem on finite groups. [6]
   b) Let G be a nontrivial finite p-group. Prove that Z(G), the centre of G is nontrivial. [4]

5. a) Find the number of elements of order 3 in a nonabelian group of order 39. [5]
   b) Prove that a group of order 36 is not simple. [5]

6. a) Assuming the result— 'R[x] is a commutative ring with identity; when R is a commutative ring with identity' prove that R[x] is an integral domain if R is an integral domain. Also prove that R and R[x] have the same characteristic. [3+1]
   b) Find all maximal ideals of $\dfrac{\mathbb{R}[X]}{(X^4-1)}$. [3]
   c) State and prove 3rd isomorphism theorem for rings. [3]

7. a) Prove that order of a finite field is p^n for some prime p and $n \geq 1$. [3]
   b) Let F be a field and $\alpha : F \to F$ be a ring homomorphism. Prove that α is 1 – 1. Is α always an isomorphism? [3]
   c) In a commutative ring with identity, prove that every proper ideal can be extended to a maximal ideal. [4]

8. a) Prove that in a principal ideal domain every nonzero prime ideal is maximal. [3]
   b) Prove that in a unique factorization domain an irreducible element is prime. [3]
   c) In $\mathbb{Z}[i\sqrt{5}]$, show that the element 3 is irreducible but not prime. [4]

Group - B

Answer <u>any six</u> questions :

9. a) Let $f:(\mathbb{R}^2-\{(1,1)\})\to R$ be defined by $f(x,y)=\dfrac{(x-y)^2}{1-2xy+y^2}$

Prove that $\lim\limits_{x\to1}\lim\limits_{y\to1}f(x,y)$ & $\lim\limits_{y\to1}\lim\limits_{x\to1}f(x,y)$ exist. Does $\lim\limits_{(x,y)\to(1,1)}f(x,y)$ exist? Justify your answer. [3]

b) Let $f(x,y)=\begin{cases}(ax+by)\sin\dfrac{x}{y}, y\neq 0\\ 0, y=0 \,\&\, \text{for all } x\end{cases}$

a, b are two real constants. Show that f is continuous at (0,0). [2]

10. The equation $f\left(\dfrac{x}{y},\dfrac{z}{y}\right)=0$ defines z implicitly as function of x and y, say z = g(x,y). Assume that g_x, g_y are continuous. Let the partial derivative of f with respect to second component be non-zero. Show that g(x,y) is homogeneous function of degree 1. [5]

11. Let $f:R^2\to R$ be defined by f(x,y) = $x^2\sin\dfrac{1}{x}+y^2\sin\dfrac{1}{y}$ if $xy\neq 0$

$= x^2\sin\dfrac{1}{x}$ if $x\neq 0, y=0$

$= y^2\sin\dfrac{1}{y}$ if $x=0, y\neq 0$

= 0 if (x,y) = (0,0)

Prove that $\dfrac{\partial f}{\partial x},\dfrac{\partial f}{\partial y}$ are discontinuous at (0,0) but f(x,y) is differentiable at (0,0) [5]

12. Let $f:S\to R$ where S is open subset of $\mathbb{R}^2$. Let $(a,b)\in S$. Let (i) f_x, f_y be defined in some neighbourhood of (a,b) (ii) f_{xy} be continuous at (a,b).

Prove that $f_{yx}(a,b)$ exists and $f_{xy}(a,b)=f_{yx}(a,b)$. [5]

13. Transform the equation $\dfrac{\partial^2 z}{\partial x^2}-2\dfrac{\partial^2 z}{\partial x\partial y}+\dfrac{\partial^2 z}{\partial y^2}=0$ taking u = x+y, $v={}^{y}\!/\!{}_{x}$ & $w={}^{z}\!/\!{}_{x}$ for the new function $w=w(u,v)$ & z is a twice differentiable function of x & y. [5]

14. Let f(x,y) have continuous first order partial derivatives and let the directional derivative $\dfrac{\partial f}{\partial \xi_\alpha}$ exist.

Prove that $\dfrac{\partial f}{\partial \xi_\alpha}=\cos\alpha\cdot f_x(x,y)+\sin\alpha\cdot f_y(x,y)$.

where α is the angle with positive side of x axis measured in anticlockwise direction. [5]

15. Show that if $ax^2+by^2+cz^2+2hxy+2fyz+2gzx$ can be resolved into linear factors in x, y and z, then $\begin{vmatrix}a & h & g\\ h & b & f\\ g & f & c\end{vmatrix}=0$. [5]

16. Use Taylor's theorem to show that $\lim\limits_{(x,y)\to(0,0)}\dfrac{\sin xy+xe^x-y}{x\cos y+\sin 2y}=-2$ when (x,y) approach (0,0) along y = –x. [5]

17. If xyz = abc, a > 0, b > 0, c > 0; using Lagrange's method, evaluate the extreme value of bcx + cay + abz. [5]

Answer <u>any four</u> questions :

18. Let $f:[a,b]\to\mathbb{R}$ be integrable on [a,b]. If there exists a positive real number K such that $f(x)\geq K$ for all $x\in[a,b]$ then show that $\frac{1}{f}$ is also integrable on [a,b]. [5]

19. Let a function $f:[a,b]\to\mathbb{R}$ be integrable on [a,b] & $f(x)\geq 0$ for all $x\in[a,b]$. Let there exist a point c in [a,b] such that f is continuous at c and f(c) > 0, then show that $\int_a^b f > 0$. [5]

20. a) State fundamental theorem of integral calculus. [1]

    b) Prove that $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \frac{\pi}{6}$. [4]

21. a) Let $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ be both integrable on [a,b]. If g(x) has the same sign for all $x\in[a,b]$ then show that there is a number μ such that $\int_a^b f(x)g(x)dx = \mu\int_a^b g(x)dx$ where $\inf_{x\in[a,b]} f(x) \leq \mu \leq \sup_{x\in[a,b]} f(x)$. [3]

    b) Prove that $\left|\int_a^b \frac{\sin x}{x}dx\right| < \frac{4}{a}$ where $0<a<b<\infty$. [2]

22. a) Define a set of measure zero. [1]

    b) Show that the following conclusions are false :

       i) If f is Riemann integrable over [a,b] it has a primitive in [a,b] [2]

       ii) If f has a primitive in [a,b] then f is integrable over [a,b] [2]

23. a) Define function of bounded variation. [2]

    b) Show that the function $f:[0,1]\to\mathbb{R}$ defined by

$$f(x) = x^2\sin\frac{\pi}{x^2},\ x\neq 0$$
$$= 0,\qquad x = 0$$

is not a function of bounded variation on [0,1]. [3]

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